Nonphysical predictions of void fractions leads to solution divergence

First-order wave equation: FOM, unconstrained and constrained ROM; \( m = 4, N = 100 \)

where \( a \in \mathbb{R}^m, B \in \mathbb{R}^{m \times m} \) and \( B_{ij} = c \left( \int_0^1 \phi_i \phi_j dx \right) \).

Using an implicit time integration scheme

\[
(I + \Delta t \ B)a^{n+1} - a^n = 0, \quad a^n := a(t^n)
\]

or

\[
C_2^{n+1} - a^n = 0
\]

Karush-Kuhn-Tucker Condition

Because of the initial condition, \( u \geq 0 \) always.

Using Karush-Kuhn-Tucker condition, the non-negativity requirement is

\[
\lambda^T \Phi_2^{n+1} \geq 0
\]

where \( \Phi = [\phi_1, \ldots, \phi_m] \), \( \Phi \in \mathbb{R}^{N \times m} \) - matrix of POD modes

\( N \) - number of spatial points

\( \lambda \) - vector of Lagrange multipliers, \( \lambda \in \mathbb{R}^N \).

Minimize the functional

\[
J = ||C_2^{n+1} - a^n||^2 + \lambda^T \Phi_2^{n+1}
\]

which requires that

\[
J_{a^{n+1}} = 2C_2^{n+1} - 2a^n + \Phi^T \lambda = 0
\]

\[
J_{\lambda} = \Phi_2^{n+1} = 0
\]

Obtain time coefficients and Lagrange multipliers from

\[
\begin{bmatrix}
2C_2 \\
\Phi^T
\end{bmatrix}
\begin{bmatrix}
a^{n+1} \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
2a^n \\
0
\end{bmatrix}
\]

Terms of Differential Entropy Inequality

Assume \( f(x) = x^2 \) and introduce \( w \) two errors in the model:

1. \( f(x) = x^2 \) replaced by \( x^2 - 10^{-4} \);
2. POD basis functions perturbed by \( 10^{-4} \).

Nonlinear Problem

Burgers equation

\[
u_t + u \cdot \nabla u = 0, \quad x \in [0, 1]
\]

Using the POD approximation and Galerkin projection yields

\[
\hat{a}_k + a_0 a_k G_{ijk} = 0, \quad i, j, k = 1, m
\]

where \( G_{ijk} = \int_0^1 \phi_i \phi_j \phi_k dx \). For two POD modes, that is, \( m = 2 \), the discretized form of (1) becomes:

\[
\begin{bmatrix}
1 + a_0^2 G_{11} \Delta t \\
a_0^2 G_{12} \Delta t
\end{bmatrix}
\begin{bmatrix}
a^{n+1}_1 \\
a^{n+1}_2
\end{bmatrix}
= \begin{bmatrix}
a^0_1 \\
a^0_2
\end{bmatrix}
\]

\[
C_{nt} \equiv a^{n+1}_2 - a^n = 0
\]

\[
C_{nt} = \begin{bmatrix}
1 + a_0^2 G_{11} \Delta t \\
a_0^2 G_{12} \Delta t
\end{bmatrix}
\begin{bmatrix}
a^{n+1}_1 \\
a^{n+1}_2
\end{bmatrix}
= \begin{bmatrix}
a^0_1 \\
a^0_2
\end{bmatrix}
\]

Constrain time coefficients of gas void fraction such that the POD reconstructed gas void fraction, \( \epsilon \in [0.38, 1] \).